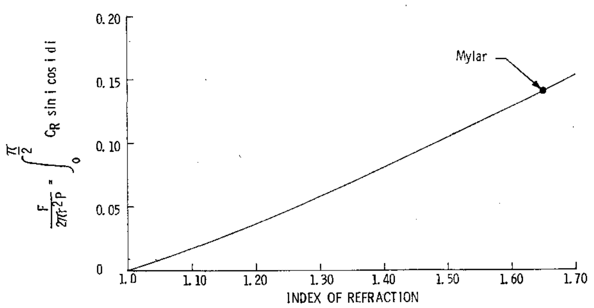

Fig. 3 Integral from Eq. (6) plotted against index of refraction.

Summation of the fractions gives: the amount reflected (R),

$$R = r + (1 - r)^2(r + r^3 + r^5 + \ldots) = 2 - 2\sum_{n=0}^{\infty}(-r)^n = \frac{2r}{1+r} \quad (2)$$

the amount transmitted (T),

$$T = (1 - r)^2(1 + r^2 + r^4 + r^6 + \ldots) = -1 + 2\sum_{n=0}^{\infty}(-r)^n = \frac{1-r}{1+r} \quad (3)$$

and the component parallel to incident ray (C),

$$C = r(1 + \cos 2i) + [\cos(i - t) + \cos(i + t)](1 - r)(r - r^2 + r^3 - r^4 + \ldots)$$
$$= 2r\cos^2 i + 2\cos i \cos t \, r(1 - r)\sum_{n=0}^{\infty}(-r)^n \quad (4)$$
$$= 2r\cos i(\cos i + [(1 - r)/(1 + r)]\cos t)$$

At point B of Fig. 1 the angles of incidence and transmittance are the same as at point A if the balloon is spherical, but the amount of light is smaller and equal to that transmitted through point A. The component parallel to the incident ray is CT. At point C, the incident light is that reflected from point B and its component parallel to the original incident ray at point A is $-CTR\cos 2i$. Similar reasoning at subsequent points yields the following:

$$C_R = C + CT(1 - R\cos 2i + R^2\cos 4i - R^3\cos 6i + \ldots) \quad (5)$$

where C_R is the total reflected component from a ray incident at point A of Fig. 1.

If the ray striking point A is considered to be a pencil of rays of cross-sectional area $dA \cos i$ striking an area dA on the balloon, the total force on the balloon F in the direction of the incident ray is

$$F = 2\pi r^2 P \int_0^{\pi/2} C_R \sin i \cos i \, d\,i \quad (6)$$

where P is the solar pressure 9.40×10^{-8} lb/ft^2, and r is the radius of the balloon satellite. This is the final equation. Figure 3 shows the integral plotted against the index of refraction for a range of refractive indices covering most transparent plastics.

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Correlation of Two-Dimensional and Axisymmetric Boundary-Layer Flows for Purely Viscous Non-Newtonian Fluids

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For steady flows in Newtonian fluids, Mangler,[1] Stepanov,[2] and Hatanaka[3] independently showed that the equations for the axisymmetric laminar boundary layer can be reduced to the equations for the two-dimensional laminar boundary layer by the transformations

$$\bar{x} = \int_0^x r^2(x)dx \qquad \bar{y} = r(x)y$$
$$r(x)\bar{v} = v + \frac{1}{r^2}\frac{dr}{dx}\bar{y}u \quad (1)$$

where x is the distance along the wall from the forward stagnation point, y is the distance from the wall, u and v are components of the velocity in the directions of x and y, respectively, and r is the distance from the axis to the wall.

The purpose of the present note is to show that, for steady flows in purely viscous non-Newtonian fluids, the equations for the axisymmetric laminar boundary layer also can be reduced to those for the two-dimensional laminar boundary layer by generalized transformations.

The equations for the steady axisymmetric boundary layer for purely viscous incompressible non-Newtonian fluids can be written as

$$(\partial/\partial x)(ru) + (\partial/\partial y)(rv) = 0 \quad (2)$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial \tau_{yx}}{\partial y} \quad (3)$$

where ρ is the density, p the pressure, and τ_{yx} is the shear stress in the x direction because of the velocity gradient in the y direction. From Bernoulli's theorem, we have

$$-(1/\rho)(dp/dx) = U(dU/dx) \quad (4)$$

It has been shown by Schowalter[4] that, for two-dimensional flows, shear stress can be expressed as

$$\tau_{yx} = a\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \qquad (n > 0) \quad (5)$$

where a and n are constants, provided that the Ostwald-de Waele (power-law) model is chosen. It can be shown easily that this equation is also valid for axisymmetric flows. Putting Eqs. (4) and (5) into Eq. (3), we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial}{\partial y}\left\{\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right\} \quad (6)$$

where $\nu = a/\rho$.

After the transformations

$$\bar{x} = \int_0^x r^{n+1}dx \qquad \bar{y} = ry$$
$$r^n\bar{v} = v + \frac{1}{r^2}\frac{dr}{dx}\bar{y}u \quad (7)$$

Eqs. (2) and (6) are reduced to

$$\partial u/\partial\bar{x} + \partial\bar{v}/\partial\bar{y} = 0 \quad (8)$$

$$u\frac{\partial u}{\partial \bar{x}} + \bar{v}\frac{\partial u}{\partial \bar{y}} = U\frac{dU}{d\bar{x}} + \nu\frac{\partial}{\partial \bar{y}}\left\{\left|\frac{\partial u}{\partial \bar{y}}\right|^{n-1}\frac{\partial u}{\partial \bar{y}}\right\} \quad (9)$$

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respectively. Equations (8) and (9) are identical with those for a two-dimensional boundary layer. It is evident that transformations (7) reduce to the corresponding expressions (1) for Newtonian fluids when $n = 1$.

It is well known that similar boundary layers exist for steady two-dimensional flows.[4] Since the equations for axisymmetric boundary layers reduce to those for two-dimensional boundary layers by the transformations (7), it is suggested that similar boundary layers exist also for steady axisymmetric flows. It will be shown in a separate paper[5] that this is true.

After the completion of this work, it was brought to the author's attention that Acrivos et al. have derived the same transformation in an unpublished paper.[6] Since, however, they confine their considerations to the cases where the velocity gradient in the boundary layer is always positive, the author believes that it would be worthwhile to present his own results.

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Equilibrium Equation for Radially Loaded Thin Circular Rings

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CONSIDER, in Fig. 1, an element ab of a thin circular ring of uniform cross section, symmetrical with respect to the plane of curvature, and subjected to distributed radial loading in this plane. Let R denote the radius of the centroidal axis of the ring, and $q(\theta)$ the intensity of radial load referred to this axis. On any cross section defined by θ, let N denote the normal force; Q, the shear force; and M, the bending moment; all considered positive in the directions shown.

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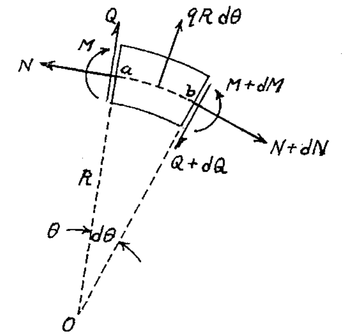

Fig. 1 Circular ring element with radical loading.

Summing forces in the radial direction $a0$ and neglecting small quantities of second order gives

$$qR\,d\theta - dQ - N\,d\theta = 0$$

from which

$$dQ/d\theta = qR - N \tag{1}$$

Summing moments about point 0 gives

$$dM = R\,dN \tag{2}$$

whereas summing moments about point b gives

$$dM/d\theta = QR \tag{3}$$

Elimination of Q and N from the equilibrium equations (1–3) gives now the third-order differential equation

$$(d^3M/d\theta^3) + (dM/d\theta) = R^2\,dq/d\theta \tag{4}$$

This expression can be integrated once to give

$$(d^2M/d\theta^2) + M = R^2\,q(\theta) + C \tag{5}$$

where C is a constant. If $q(\theta) = 0$, Eq. (5) gives $M =$ const. This simply means that any arbitrary uniform bending moment in the ring can be superimposed on that produced by the radial loading $q(\theta)$. Considering only bending caused by the q loading, we take $C = 0$ and write Eq. (5) in the form

$$(d^2M/d\theta^2) + M = R^2q(\theta) \tag{6}$$

Taking $s = R\theta$ as a new independent variable, Eq. (6) becomes

$$(d^2M/ds^2) + (M/R^2) = q(s) \tag{6a}$$

which has the same form as the familiar equation

$$(d^2w/ds^2) + (w/R^2) = M/EI \tag{7}$$

for radial deflection w of a circular ring. The existence of Eq. (6), analogous to Eq. (7) in form, does not seem to have been generally recognized. When $R = \infty$, both Eq. (6a) and Eq. (7) coincide with those for a straight bar.

Equation (6) can be useful in predicting buckling loads for uniformly compressed circular rings. When such a ring undergoes an inextensional bending during buckling, there will be a small change in curvature of the ring, as defined by the left-hand side of Eq. (7). As a result of this change in curvature, the existing normal forces N acting on each ele-

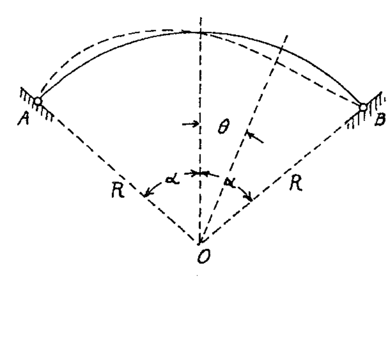

Fig. 2 Inextensional buckling of circular ring sector.